ON THE APPLICATION OF GENERAL VARIATIONAL PRINCIPLES IN THE RELATIVISTIC MECHANICS OF AN IDEAL FLUID

(O PRIMENENII OBSHCHIKH VARIATSIONNYKH PRINTSIPOV V RELIATIVISTSKOI MEKHANIKE IDEAL'NOI ZHIDKOSTI)

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Equations describing the motion of a continuous conducting medium in prescribed electromagnetic and gravitational fields are obtained most simply by variation of the appropriate Lagrangians.

As a result, the variational principle leads to a second order scalar equation; this affoeds significant advantages in the analysis of the motion of a medium as compared with the analysis of the customary equations of momentum, mass and energy conservation.

The Lagrangian of an electromagnetic field is well known to have the form [1]:

$$L_e = -F_{ik}F^{ik} / 16\pi, \qquad F_{ik} = \partial A_k / \partial x^i - \partial A_i / \partial x^k \tag{0.1}$$

Here F_{ik} and A_i are the tensor and quadri-potential of the electromagnetic field.

We have for the Lagrangian of a continuum in the case of isentropic quasi-potential flow [2]

$$L_m = p = (w - E) / v = [c\sqrt{-g^{ik}S_iS_k} - E] / v \tag{0.2}$$

Here S is the effect for matter, E ise mass energy density, v the specific volume, p the pressure, w the heat content, g_{ik} the metric tensor components; furthermore

$$S_i = \partial S / \partial x^i, \qquad cS_i = wu_i \tag{0.3}$$

where c is the velocity of light, u_i the quadri-velocity.

We will seek the equation of motion for the established form of the Lagrangian for a continuum exactly as for the field

$$\frac{\partial}{\partial S}(\sqrt{-g}L_m) - \frac{\partial}{\partial x^k}\frac{\partial(\sqrt{-g}L_m)}{\partial S_k} = 0 \tag{0.4}$$

where g is the determinant of the metric tensor g_{ik}. In the case when an electromagnetic field is present, it is necessary to replace cS_i by $wu_i + (e/m)A_i$, after the evaluation of the derivatives, where (e/m) is the mean ratio of the particle charge to the mass.

If there is no electromagnetic field, we obtain the continuity equation

$$\frac{\partial}{\partial x^k}\frac{\sqrt{-g}S^k}{vw} = 0 \qquad \text{or} \qquad \frac{\partial}{\partial x^k}\frac{\sqrt{-g}u^k}{v} = 0 \tag{0.5}$$

by substituting (0.2) into (0.4).

Since $v \sim (w - \alpha c^2)^{-1/(k-1)}$ for $pv^k = \text{const}$, then (0.2) may be written as

$$L_m = p = \text{const}\left[-\alpha c + \sqrt{-g^{ik}S_i S_k}\right]^{\frac{k}{k-1}} \tag{0.6}$$

Analogously, a more complex equation is hence obtained from (0.4)

$$\frac{\partial}{\partial x^n}\left[\frac{\sqrt{-g}S^n}{\sqrt{-S_l S^l}}\left(\sqrt{-S_l S^l} - \alpha c\right)^{\frac{1}{k-1}}\right] = 0 \tag{0.7}$$

However, it is convenient to use the sound equation instead of (0.7). To obtain this latter, let us write (0.5) as

$$\frac{S^n}{vw}\frac{\partial \ln \sqrt{-g}}{\partial x^n} + \frac{\partial (S^n / vw)}{\partial x^n} = 0 \tag{0.8}$$

Since

$$\frac{d \ln w}{d \ln v} = -\frac{\omega^2}{c^2} \tag{0.9}$$

where ω is the relativistic velocity of sound, we then find from (0.8)

$$2\frac{\omega^2}{c^2}(S_n S^n)\left(S_l{}^l + S^l\frac{\partial \ln \sqrt{-g}}{\partial x^l}\right) + \left(1 - \frac{\omega^2}{c^2}\right)S^l\left(\frac{\partial S_n}{\partial x^l}S^n + \frac{\partial S^n}{\partial x^l}S_n\right) = 0 \tag{0.10}$$

Equation (0.10) will be the original in the analysis of different examples of the motion of a relativ stic medium in the special theory of relativity (Section 1) and the spherically symmetrical motion in a Schwarzschild gravitational field (Section 2). It is investigated by the method of characteristics.

It is convenient to use (0.10) even in the absence of a gravitational field if the calculations are carried out in any curvilinear coordinate system.

1. As examples of the utilization of (0.10), let us consiser one-dimensional nonstationary waves (Riemann waves) and two-dimensional plane stationary gas flow.

Let us use a Galilean metric for the mentioned problems

$$ds^2 = c^2dt^2 - (dx^\alpha)^2, \qquad -g_{00} = g_{11} = g_{22} = g_{33} = 1 \qquad (\alpha = 1, 2, 3) \tag{1.1}$$

For the chosen metric $\partial \ln \sqrt{-g} / \partial x^l = 0$ and from (0.10) we have

$$2\frac{\omega^2}{c^2}S_l{}^l(S_k S^k) + \left(1 - \frac{\omega^2}{c^2}\right)S^l\left[\frac{\partial S_k}{\partial x^l}S^k + \frac{\partial S^k}{\partial x^l}S_k\right] = 0 \tag{1.2}$$

In the one-dimensional unsteady flow case $l, k = 0, 1$, in which $S_1 = S^1$ and $S_0 = -S^0$ for the chosen metric. Hence, after elementary manipulations, we find from (1.2)

$$\frac{\omega^2}{c^2}(S_{11} - S_{00})(S_1{}^2 - S_0{}^2) + \left(1 - \frac{\omega^2}{c^2}\right)(S_0{}^2S_{00} - 2S_0S_1S_{01} + S_1{}^2S_{11}) = 0$$

Since

$$S_{11} = \partial S_1 / \partial x, \qquad S_{00} = \partial S_0 / \partial \tau, \qquad \tau = ct$$

we then find from the latter equation

$$\frac{\omega^2}{c^2}\left(\frac{\partial S_1}{\partial x}-\frac{\partial S_0}{\partial \tau}\right)(S_1^2-S_0^2)+\left(1-\frac{\omega^2}{c^2}\right)\left(S_0^2\frac{\partial S_0}{\partial \tau}-2S_0S_1\frac{\partial S_1}{\partial \tau}+S_1^2\frac{\partial S_1}{\partial x}\right)=0 \quad (1.3)$$

Using a known method of change of variables, we can transform to a linear equation from (1.3). To do this, let us divide (1.3) by the Jacobian $\partial (S_0, S_1) / \partial (\tau, x)$, by first assuming it to be different from zero.

Hence we obtain (1.4)

$$\frac{\omega^2}{c^2}\left(\frac{\partial \tau}{\partial S_0}-\frac{\partial x}{\partial S_1}\right)(S_1^2-S_0^2)+\left(1-\frac{\omega^2}{c^2}\right)\left(S_0^2\frac{\partial x}{\partial S_1}+2S_0S_1\frac{\partial \tau}{\partial S_1}+S_1^2\frac{\partial \tau}{\partial S_0}\right)=0$$

The effect S is a quasi-potential [2], hence $\partial S_0 / \partial x = \partial S_1 / \partial \tau$. This equality may be written in the form of an equality of Jacobians, which will yield after having been divided by $\partial (S_0, S_1) / \partial (\tau, x)$

$$-\frac{\partial \tau}{\partial S_1}=\frac{\partial x}{\partial S_0} \quad (1.5)$$

For the sequel it is expedient to introduce a function Ψ such that

$$\tau = \partial \Psi / \partial S_0 = \Psi_0, \qquad x = \partial \Psi / \partial S_1 = \Psi_1 \quad (1.6)$$

Then (1.4) will become (1.7)

$$\frac{\omega^2}{c^2}(\Psi_{00}-\Psi_{11})(S_1^2-S_0^2)+\left(1-\frac{\omega^2}{c^2}\right)(S_0^2\Psi_{11}+S_1^2\Psi_{00}+2S_0S_1\Psi_{01})=0$$

In case the Jacobian $\partial (S_0, S_1) / \partial (\tau, x)$ equals zero, the change of variables in (1.3) is impossible. However, the very fact that the Jacobian is zeo means that $S_0 = f(S_1)$. Hence

$$\frac{\partial (S_1, S_0)}{\partial (x, \tau)}=\frac{\partial S_1}{\partial x}\frac{\partial S_0}{\partial \tau}-\frac{\partial S_1}{\partial \tau}\frac{\partial S_0}{\partial x}=0$$

which becomes, by virtue of the equality $\partial S_0 / \partial x = \partial S_1 / \partial \tau$

$$\frac{dS_0}{dS_1}\frac{\partial S_1}{\partial x}-\frac{\partial S_1}{\partial \tau}=0 \quad (1.8)$$

The solution of the latter equation is

$$x + \tau dS_0 / dS_1 = F(S_1) \quad (1.9)$$

where $F(S_1)$ is an arbitrary function of S_1 determined from the boundary conditions. In order to determine the form of the function $S_0 = f(S_1)$, let us use the equalities

$$\frac{\partial S_0}{\partial x}=\frac{dS_0}{dS_1}\frac{\partial S_1}{\partial x}, \qquad \frac{\partial S_0}{\partial \tau}=\frac{dS_0}{dS_1}\frac{\partial S_1}{\partial \tau}$$

to help us find from (1.3)

$$\frac{\partial S_1}{\partial x}\left[\frac{\omega^2}{c^2}(S_1^2-S_0^2)+\left(1-\frac{\omega^2}{c^2}\right)S_1^2\right]+\frac{\partial S_1}{\partial \tau}\left[\left(1-\frac{\omega^2}{c^2}\right)S_0^2\frac{dS_0}{dS_1}-\right.$$
$$\left.-\frac{\omega^2}{c^2}\frac{dS_0}{dS_1}(S_1^2-S_0^2)-2S_0S_1\left(1-\frac{\omega^2}{c^2}\right)\right]=0$$

Hence, taking (1.8) into account, we obtain an equation to determine S_0 as a function of S_1

$$\left(\frac{dS_0}{dS_1}\right)^2\left(S_1{}^2 - S_0{}^2\,\frac{c^2}{\omega^2}\right) + 2\left(\frac{c^2}{\omega^2} - 1\right) S_1 S_0\,\frac{dS_0}{dS_1} + \left(S_0{}^2 - \frac{c^2}{\omega^2}\,S_1{}^2\right) = 0 \qquad (1.10)$$

Before solving (1.10), let us examine the equation of plane stationary flow of a relativistic gas. In form, this latter will agree with the equations for one-dimensional unsteady flow. Writing these equations jointly, let us seek the general solution of the problems under consideration.

As is known, the action function in the case of stationary flow may be written thus

$$S = -w_0 t + S\,(x^\alpha) \qquad (1.11)$$

Here $a_{1,2}$ are the components of the conventional velocity

$$cS_0 = -w_0 = -w\,/\,\theta, \qquad S_1 = w_0\,a_1\,/\,c^2, \qquad S_2 = w_0 a_2\,/\,c^2$$

$$\theta = \sqrt{1 - a^2\,/\,c^2} \qquad (1.12)$$

Substituting (1.11) into (1.2) and taking into account that (see (1.12))

$$S_0{}^\circ = -S_{00} = 0, \qquad S_1 = S^1,\ S_2 = S^2, \qquad S_0 S^\circ = -S_0{}^2 = -w_0{}^2\,/\,c^2$$

we find the equation for plane stationary flow

$$\frac{\omega^2}{c^2}\left(\frac{\partial S_1}{\partial x} + \frac{\partial S_2}{\partial y}\right)\left(S_1{}^2 + S_2{}^2 - \frac{w_0{}^2}{c^2}\right) +$$
$$+\left(1 - \frac{\omega^2}{c^2}\right)\left(\frac{\partial S_1}{\partial x}\,S_1{}^2 + \frac{\partial S_2}{\partial y}\,S_2{}^2 + 2S_1S_2\,\frac{\partial S_2}{\partial x}\right) = 0 \qquad (1.13)$$

Here we use the equality

$$\partial S_2\,/\,\partial x = \partial S_1\,/\,\partial y \qquad (1.14)$$

Changing the variables in (1.13) and introducing the function Ψ by virtue of (1.14) so that $x = \partial\Psi\,/\,\partial S_1 = \Psi_1,\ y = \partial\Psi\,/\,\partial S_2 = \Psi_2,$ we obtain

$$\frac{\omega^2}{c^2}\,(\Psi_{22} + \Psi_{11})\left(S_1{}^2 + S_2{}^2 - \frac{w_0{}^2}{c^2}\right) +$$
$$+\left(1 - \frac{\omega^2}{c^2}\right)(S_1{}^2\Psi_{22} + S_2{}^2\Psi_{11} - 2S_1S_2\Psi_{12}) = 0 \qquad (1.15)$$

If the Jacobian $\partial\,(S_1, S_2)\,/\,\partial\,(x, y) = 0$, the change of variables in (1.13) is impossible. Then, just as in the case considered earlier, we have

$$x + y\ dS_2\,/\,dS_1 = F\,(S_1) \qquad (1.16)$$

and the function $S_2 = f(S_1)$ is found from Equation

$$\left(\frac{dS_2}{dS_1}\right)^2\left(S_1{}^2 + \frac{c^2}{\omega^2}\,S_2{}^2 - \frac{w_0{}^2}{c^2}\right) + 2S_1S_2\left(\frac{c^2}{\omega^2} - 1\right)\frac{dS_2}{dS_1} +$$
$$+\left(S_2{}^2 + \frac{c^2}{\omega^2}\,S_1{}^2 - \frac{w_0{}^2}{c^2}\right) = 0 \qquad (1.17)$$

Now, the equations for one-dimensional unsteady (1.7) and two-dimensional stationary flow (1.15) may be written as one equation

$$\frac{\omega^2}{c^2}\,(\Psi_{\beta\beta} \pm \Psi_{11})\left(S_1{}^2 \pm S_\beta{}^2 - \frac{\beta}{2}\,\frac{w_0{}^2}{c^2}\right) +$$
$$+\left(1 - \frac{\omega^2}{c^2}\right)(S_1{}^2\Psi_{\beta\beta} + S_\beta{}^2\,\Psi_{11} \mp 2S_1S_\beta\Psi_{1\beta}) = 0 \qquad (1.18)$$

The upper signs in (1.18) correspond to plane stationary flow when $\beta = 2$, $x_\beta = y$, $S_\beta = S_2$, the lower signs to one-dimensional unsteady flow when $\beta = 0$, $x_0 = \tau$, $S_\beta = S_0$.

Furthermore, let us introduce the notation (see (1.12))

$$S_1^2 + S_2^2 = w_0^2 (a_1^2 + a_2^2) / c^4 = w_0^2 a^2 / c^4 = c^2 z^2 \qquad (1.19)$$

where $z = w_0 |a| / c^3$.

It is now expedient to introduce the following substitutions for (1.18):

$$S_1 = cz \sin\varphi, \quad S_2 = cz\cos\varphi, \qquad \frac{S_1}{S_2} = \frac{a_1}{a_2} = \tan\varphi \qquad (\beta = 2)$$

$$S_0 = -cz\cosh\varphi, \quad S_1 = cz\sinh\varphi, \qquad -\frac{S_1}{S_0} = \frac{a}{c} = \tanh\varphi, \qquad \frac{w^2}{c^2} = S_1^2 - S_0^2 = c^2 z^2 \qquad (\beta = 0) \qquad (1.20)$$

In the new variables (1.20) Equation (1.18) becomes

$$\frac{\omega^2}{c^2} z^2 \Psi_{zz} + z\Psi_z \pm \Psi_{\varphi\varphi} = \frac{\beta}{2}\frac{\omega^2}{z^2 c^2}\left(\frac{w_0}{c^2}\right)^2 (\Psi_{zz} z^2 + \Psi_z z \pm \Psi_{\varphi\varphi}) \qquad (1.21)$$

in which

$$x = \Psi_z \sin\varphi + \Psi_\varphi \cos\varphi / z, \qquad y = \Psi_z\cos\varphi - \Psi_\varphi \sin\varphi / z \quad (\beta = 2)$$

$$x = -\Psi_z \sinh\varphi + \Psi_\varphi\cosh\varphi / z, \qquad \tau = -\Psi_z\cosh\varphi + \Psi_\varphi \sinh\varphi / z \quad (\beta = 0)$$

Equation (1.21) is simplified considerably by the substitution $\xi = \ln z$

$$\frac{\omega^2}{c^2}\Psi_{\xi\xi} + \left(1 - \frac{\omega^2}{c^2}\right)\Psi_\xi \pm \Psi_{\varphi\varphi} = \frac{\beta}{2z^2}\frac{\omega^2}{c^2}\left(\frac{w_0}{c^2}\right)^2 (\Psi_{\xi\xi} \pm \Psi_{\varphi\varphi}) \qquad (1.22)$$

Equation (1.22) may be solved by the method of characteristics.

Let us now turn to seeking the singular solutions of the joint equation of the two problems under consideration. Combining (1.10) and (1.17) we obtain

$$\left(\frac{dS_\beta}{dS_1}\right)^2 \left[S_1^2 \pm \left(\frac{c^2}{\omega^2} S_\beta^2 - \frac{\beta}{2}\frac{\omega_0^2}{c^2}\right)\right] + 2S_1 S_\beta\left(\frac{c^2}{\omega^2} - 1\right)\frac{dS_\beta}{dS_1} + \left[S_\beta^2 \pm \left(\frac{c^2}{\omega^2} S_1^2 - \frac{\beta}{2}\frac{w_0^2}{c^2}\right)\right] = 0 \qquad (1.23)$$

Finding the roots of the square of the derivative dS_β / dS_1 in (1.23), we find in the case $\beta = 0$ and $cS_0 = -w / \theta$, $cS_1 = wa / c\theta$

$$-\frac{dS_0}{dS_1} = \frac{a/c \pm \omega / c}{1 \pm a\omega / c^2} \qquad (1.24)$$

Substituting the result obtained into (1.9), we have

$$x = \frac{a/c \pm \omega / c}{1 \pm a\omega / c^2} + F(a) \qquad (1.25)$$

which, as is known, is the equation of relativistic Riemann waves.

According to (1.12), we have from (1.23) for $\beta = 2$

$$\frac{dS_2}{dS_1} = \frac{da_y}{da_x} = \left\{-\frac{a_x a_y}{c^2}\left(\frac{c^2}{\omega^2} - 1\right) \pm \left[\left(1 - \frac{a^2}{c^2}\right)\left(\frac{a^2}{\omega^2} - 1\right)\right]^{1/2}\right\}\left(\frac{a_x^2}{c^2} + \frac{a_y^2}{\omega^2} - 1\right)^{-1} \qquad (1.26)$$

Here it is assumed that $|a| > \omega$

Since $\omega = \omega(w) = \omega(\theta w_0) = \omega(a)$, we can determine a_y as a function of a_x from (1.26). Substituting the found function $da_y / da_x = B(a_x)$ into (1.16), we find the solution for generalized Prandtl-Mayer flow

$$x + yB(a_x) = F(a_x) \tag{1.27}$$

In conclusion, let us show how the transition from (1.21) to the customary (nonrelativistic) equation for gas flows is accomplished.

In the $\beta = 0$ case we have from (1.21)

$$\frac{\omega^2}{c^2} z^2 \Psi_{zz} + z\Psi_z = \Psi_{\varphi\varphi} \tag{1.28}$$

In the case of nonrelativistic gas flow we have

$$\begin{aligned} z = w / c^2 = 1 + i / c^2, \qquad \varphi = \mathrm{Ar\,tanh}\, a / c \\ dz = di / c^2, \qquad d\varphi \approx d(a / c) \quad (a \ll c) \end{aligned} \tag{1.29}$$

Taking account of (1.29), let us write (1.28) as

$$\omega^2 c^2 (1 + 2i / c^2) \Psi_{ii} + (1 + i / c^2) c^2 \Psi_i = c^2 \Psi_{aa}$$

Hence, as $c \to \infty$ we have the well-known Riemann equation

$$\omega^2 \Psi_{ii} + \Psi_i = \Psi_{aa} \tag{1.30}$$

whose singular solution is

$$x = (a \pm \omega) t + F(a), \qquad da \pm \omega\, d \ln v = 0 \tag{1.31}$$

In the $\beta = 2$ case we have from (1.21)

$$\frac{\omega^2}{c^2} \Psi_{zz} z^2 + \Psi_z z + \Psi_{\varphi\varphi} = \frac{\omega^2}{z^2 c^2} \left(\frac{w_0}{c^2}\right)^2 (\Psi_{zz} z^2 + \Psi_z z + \Psi_{\varphi\varphi}) \tag{1.32}$$

Since $w_0/c^2 \approx 1$ for $a \ll c$ it then follows from (1.19) that $z \approx a/c$.

Using the mentioned limiting values for w_0 and z, we obtain the known equation describing stationary gas flow from (1.32).

$$(\Psi_a a + \Psi_{\varphi\varphi}) (1 - \omega^2 / a^2) = \omega^2 \Psi_{aa} \tag{1.33}$$

For $a \ll c$ we have from (1.26)

$$da_y / da_x = (-a_x a_y \pm \omega \sqrt{a^2 - \omega^2}) / (a_y^2 - \omega^2) \tag{1.34}$$

Substituting (1.34) into (1.27) and assuming $F(a_x) = 0$, we obtain the Prandtl-Mayer solution

$$x / y = - da_y / da_x = (a_x a_y \mp \omega \sqrt{a^2 - \omega^2}) / (a_y^2 - \omega^2) \tag{1.35}$$

2. The space-time interval in the Schwarzschild gravitational field may be written as

$$ds = \left(1 - \frac{r_0}{r}\right) c^2 dt^2 - \frac{dr^2}{1 - r_0/r} - r^2 (\sin^2\theta\, d\varphi^2 + d\theta^2) \tag{2.1}$$

i.e. the components of the metric tensor are [1]

$$\begin{aligned} g_{00} = -(1 - r_0 / r), \qquad g_{22} = r^2 \\ g_{11} = (1 - r_0 / r)^{-1}, \qquad g_{33} = r^2 \sin^2\theta \end{aligned} \tag{2.2}$$

To obtain the equation of motion, let us again use (0.10) in which it should be taken into account that

$$S_l{}^l = \frac{\partial}{\partial x^l}(g^{il}S_i)$$

then we will have

$$\frac{\omega^2}{c^2}\left[S_1{}^2\left(1-\frac{r_0}{r}\right)-\frac{S_0{}^2}{1-r_0/r}\right]\left[S_{11}\left(1-\frac{r_0}{r}\right)-\frac{S_{00}}{1-r_0/r}+\frac{2S_1}{r}\left(1-\frac{r_0}{r}\right)\right]+$$
$$+\left(1-\frac{\omega^2}{c^2}\right)\left\{S_{11}S_1\left(1-\frac{r_0}{r}\right)^2-2S_0S_1S_{10}+\frac{S_0{}^2S_{00}}{(1-r_0/r)^2}+\frac{S_1(1-r_0/r)r_0}{2r^2}\times\right.$$
$$\left.\times\left[S_1{}^2+\frac{S_0{}^2}{(1-r_0/r)^2}\right]\right\}=0 \tag{2.3}$$
$$S_{11}=\frac{\partial}{\partial r}S_1, \quad S_{00}=\frac{\partial}{\partial x^0}S_0, \quad cS_1=wu_1, \quad cS_0=wu_0$$

Here w is the relativistic heat content, u_i the quadri-velocity. The quantity r_0 is the gravitational radius of the mass producing the gravitational field. To simplify (2.3), let us introduce the new independent variable

$$d\xi = dr\,/\,(1-r_0\,/\,r) \tag{2.4}$$

Then

$$\left(1-\frac{r_0}{r}\right)S_1=S_\xi, \qquad \left(1-\frac{r_0}{r}\right)^2S_{11}=S_{\xi\xi}-\frac{r_0}{r^2}S_\xi$$

After elementary manipulations, (2.3) takes the form

$$\frac{\omega^2}{c^2}(S_\xi{}^2-S_0{}^2)(S_{\xi\xi}-S_{00})+\left(1-\frac{\omega^2}{c^2}\right)(S_{\xi\xi}S_\xi{}^2-2S_0S_\xi S_{0\xi}+S_{00}S_0{}^2)+$$
$$+\frac{2S_\xi}{r}(S_\xi{}^2-S_0{}^2)\left[\frac{\omega^2}{c^2}-\left(1+3\frac{\omega^2}{c^2}\right)\frac{r_0}{4r}\right]=0 \tag{2.5}$$

This equation may easily be investigated by using characteristics which have the form

$$\dot\xi^2\left(\frac{\omega^2}{c^2}B^2-A^2\right)-2\dot\xi AB\left(1-\frac{\omega^2}{c^2}\right)-\left(B^2-A^2\frac{\omega^2}{c^2}\right)=0 \tag{2.6}$$

and the condition on the characteristics

$$\dot A\dot\xi\left(\frac{\omega^2}{c^2}B^2-A^2\right)+D\dot\xi=\dot B\left(B^2-A^2\frac{\omega^2}{c^2}\right) \tag{2.7}$$

Here

$$A=S_0, \qquad B=S_\xi, \qquad D=\frac{2B}{r}(B^2-A^2)\left[\left(1+3\frac{\omega^2}{c^2}\right)\frac{r_0}{4r}-\frac{\omega^2}{c^2}\right]$$

and the dot denotes the total derivative with respect to time $x^0 = ct$.

Let us transform (2.6) and (2.7) to a form similar to the analogous expressions for the characteristics and the condition on them in the special theory of relativity.

To do this, let us write A and B as

$$A=S_0=g_{00}u^0w\,/\,c=-w^*\,/\,c\theta$$
$$B=S_\xi=(1-r_0\,/\,r)\,S_1=(1-r_0\,/\,r)\,g_{11}wu^1\,/\,c=w^*a\,/\,c^2\theta \tag{2.8}$$

Here

$$\theta = \sqrt{1 - a^2 / c^2}, \qquad w^* = \sqrt{1 - r_0 / r}\, w, \qquad a = \sqrt{v_1 v^1} \qquad (2.9)$$

and v^1 is the conventional velocity measured in intrinsic time [1]. Using (2.8) we find from the equation of the characteristics (2.6)

$$\frac{d\xi}{dx^\circ} = \frac{a / c \pm \omega / c}{1 \pm \omega a / c^2} \qquad (2.10)$$

The fundamental effect of the approximation in the Schwarzschild sphere is seen from the obtained relationship. In fact, for gas moving to the center ($a \to -a$) we find from (2.10)

$$\xi - \xi' = (r - r') + r_0 \ln\left(\frac{r - r_0}{r' - r_0}\right) = -\int_{x_0'}^{x_0} \frac{a / c \pm \omega / c}{1 \pm \omega a / c^2}\, dx^\circ$$

where r' is the value of the coordinate at the time $x_0' = ct'$.

Because of the finiteness of the integrand, we have that

$$t \approx -\frac{r_0}{c} \ln\left|\frac{r}{r_0} - 1\right| \to \infty \qquad \text{for } r \to r_0$$

This latter means that any perturbation being propagated along characteristics reaches the Schwarzchild sphere in a time which is infinite for the external observer.

Using (2.8) to (2.10) the condition on the characteristics (2.7) becomes

$$\frac{d}{dt} \ln w^* - \frac{2a}{(1 \pm \omega a/c^2)\, r}\left[\left(1 + 3\frac{\omega^2}{c^2}\right)\frac{r_0}{4r} - \frac{\omega^2}{c^2}\right] \pm \frac{1}{\theta^2}\frac{\omega}{c}\frac{d}{dt}\left(\frac{a}{c}\right) = 0 \qquad (2.11)$$

These conditions hold along the characteristics (2.10)

Hence, the solution of the equations describing the gas motion is not difficult by the method of characteristics in the Schwarzchild field.

By passing to the limit in (2.10) and (2.11) we arrive at the equation of motion of a nonrelativistic gas in a gravity field. To do this, let us note that

$$\frac{d \ln w^*}{dt} = \frac{d \ln w}{dt} + \frac{1}{2}\frac{d \ln (1 - r_0 / r)}{dt}, \qquad \frac{d \ln w}{a \ln v} = -\frac{\omega^2}{c^2}$$

Then the equation of the characteristics (2.10) takes the form $\dot{r} = a \pm \omega$ and the condition on the characteristics becomes [3]

$$da \pm (2 a\omega\, dt / r - \omega\, d \ln v) - g\, dt = 0, \qquad g = -kM / r^2$$

Here v is the specific volume; ω is the velocity of sound and g the acceleration due to gravity.

BIBLIOGRAPHY

1. Landau, L.D. and Lifshits, E.M., Teoriia polia (Field Theory). Fizmatgiz, 1960.
2. Staniukovich, K.P., Lagranzhian sploshnoi sredy v rimanovom prostranstve (Lagrangian of a continuum in Riemann space). Dokl.Akad.Nauk SSSR, Vol. 154, № 2, 1964.
3. Staniukovich, K.P., Neustanovivshiesia dvizheniia sploshnoi sredy (Unsteady Motion of a Continuum). Gostekhizdat, 1955.

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